

AVERAGING SEQUENCES AND ABELIAN RANK IN AMENABLE GROUPS

BY

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ABSTRACT

We investigate the connection between the abelian rank of a countable amenable group and the existence of good averaging sequences (e.g., for the ergodic theorem). We show that if G is a group with finite abelian rank $r(G)$, then $2^{r(G)}$ is a lower bound on the constant associated to a Tempel'man sequence, and if G is abelian there is a Tempel'man sequence in G with this constant. On the other hand, infinite rank precludes the existence of Tempel'man sequences and forces all tempered sequences to grow super-exponentially.

1. Introduction

A countable group G is **amenable** if there exists a sequence $\{F_n\}$ of finite subsets of G with the property that for every $g \in G$,

$$\frac{1}{|F_n|} |gF_n \cap F_n| \rightarrow 1.$$

(See [3] for some equivalent definitions.) Such a sequence is called a **Følner sequence**. For example, $F_n = \{0, \dots, n\} \subseteq \mathbb{Z}$ satisfy this condition.

Much of ergodic theory — i.e., the study of the dynamics of measure-preserving transformations — can be extended to measure-preserving actions of amenable groups. A good example of this, which explains why Følner sequences are sometimes called averaging sequences, is the mean ergodic theorem: given

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a measure preserving action $(g, \omega) \mapsto g\omega$ of G on a probability space $(\Omega, \mathcal{B}, \mu)$, for any $\varphi \in L^2(\Omega, \mathcal{B}, \mu)$ and any Følner sequence $\{F_n\}$,

$$(1.1) \quad \frac{1}{|F_n|} \sum_{g \in F_n} \varphi^g \rightarrow \mathbb{E}(\varphi | \mathcal{B}_0)$$

in L^2 . Here $\mathcal{B}_0 \subseteq \mathcal{B}$ is the σ -algebra of G -invariant measurable sets, and $\varphi^g(x) = \varphi(gx)$. F. Riesz's proof of the mean ergodic theorem for \mathbb{Z} -actions carries over almost verbatim to the general case.

However, for some results — in particular, for many pointwise results — it is necessary to place some restrictions on the averaging sequence $\{F_n\}$. One such condition is the following, which requires that the sets F_n be invariant to some degree to translation by their own elements. As usual, we write $AB = \{ab : a \in A, b \in B\}$ and $A^{-1} = \{a^{-1} : a \in A\}$.

Definition 1.1: An increasing Følner sequence $\{F_n\}$ in G is called a **Tempel'man sequence** if there is a constant C such that for every n , $|F_n^{-1}F_n| \leq C|F_n|$

Using Tempel'man sequences in the role of averaging sequence one can prove the pointwise ergodic theorem (Birkhoff's theorem), which states that the limit (1.1) converges almost surely for $\varphi \in L^1$ [7], and the Shannon–McMillan–Breiman theorem [5].

Two classes of groups which possess Tempel'man sequences are the locally finite groups and the finitely generated groups with polynomial growth. For a long time no examples were known of amenable groups without Tempel'man sequences. Recently, E. Lindenstrauss [4] showed that there do exist such groups, demonstrating that the lamplighter group L is one such group. This group is defined as follows: let \mathbb{Z} act on $V = \bigoplus_{i=-\infty}^{\infty} (\mathbb{Z}/2\mathbb{Z})$ by coordinate shift, and set $L = \mathbb{Z} \ltimes V$.

In this paper we investigate the connection between the abelian rank of a group and the behavior of the averaging sequences it contains. Rank is defined as follows:

Definition 1.2: The **abelian rank** of a group G is

$$r(G) = \sup\{n : G \text{ contains a subgroup isomorphic to } \mathbb{Z}^n\}.$$

Our first result connects abelian rank with Tempel'man sequences:

THEOREM: *Let G be a countable group.*

- (1) *If $r(G) < \infty$, then $2^{r(G)}$ is a lower bound on the constant associated with a Tempel'man sequence in G . If in addition G is abelian, then there exists in G a Tempel'man sequence with constant $2^{r(G)}$.*
- (2) *If $r(G) = \infty$, then G has no Tempel'man sequences.*

This provides many new examples of groups without Tempel'man sequences, for instance $\bigoplus_{i=-\infty}^{\infty} \mathbb{Z}$.

With regard to the problem of the existence of good averaging sequences, Lindenstrauss demonstrated in [4] that tempered sequences, defined below, can serve as averaging sequences for the pointwise ergodic theorem and the SMB theorem:

Definition 1.3: A Følner sequence $\{F_n\}$ is **tempered** if there exists a constant C such that $|(\bigcup_{i < n} F_i^{-1})F_n| \leq C|F_n|$ for every n .

Note that every Tempel'man sequence is tempered.

Tempered sequences have the advantage that they exist in every amenable group; in fact, every Følner sequence has a tempered subsequence. However, tempered sequences may grow quickly. For example, every tempered sequence in the lamplighter group grows super-exponentially [4]. Fast growth implies, for instance, that in order to estimate the mean of a process from observations, the size of the data set needed can grow sharply from one estimate to the next. In contrast, when calculating the mean of a \mathbb{Z} -process using a running average, each additional symbol of the output can be used to update the estimate.

Our second result shows that abelian rank is relevant to this question as well:

THEOREM: *Let G be a countable group. If $r(G) = \infty$ every tempered sequence in G grows super-exponentially.*

These results leave open the class of non-abelian finite rank groups, which contains groups both with and without “good” averaging sequences. We will say a little about this at the end of section 4.

Our main tool is the observation that the Brunn–Minkowski inequality, which bounds the volumes of sums of sets in \mathbb{R}^d , can be used to bound the sums of sufficiently invariant sets in \mathbb{Z}^d . This is developed in the next section. In section 3 we derive the results about Tempel'man sequences. Section 4 addresses the growth of tempered sequences.

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2. Tempel'man sequences in integer lattices

We will need the following classical theorem (see, e.g., [1]):

THEOREM (Brunn–Minkowsky inequality): *Let $A, B \subseteq \mathbb{R}^d$ be measurable sets such that $A + B$ is also measurable. Then, denoting the d -dimensional Lebesgue measure by ν ,*

$$\nu(A + B) \geq (\nu(A)^{1/d} + \nu(B)^{1/d})^d.$$

We will use this to obtain a similar statement for sets in \mathbb{Z}^d when one of the sets is sufficiently self-invariant, with counting measure replacing Lebesgue measure. Note that in general the re-statement of the Brunn–Minkowsky for \mathbb{Z}^d is false. This is demonstrated by considering the “simplex” $A = B = \{0, e_1, \dots, e_d\}$ with e_i the standard generators of \mathbb{Z}^d .

We note that a variant of the Brunn–Minkowsky inequality for integer lattices was obtained by R. J. Gardner and P. Gronchi in [2]. Also note that I. Z. Ruzsa [6] has obtained inequalities for sums of sets in \mathbb{Z}^d which are sufficient for the qualitative results stated below. However, the tight bound below does not follow from that work.

Denote by $e_i \in \mathbb{Z}^d$ the standard generators of \mathbb{Z}^d , that is, $(e_i)_j = \delta_{i,j}$.

THEOREM 2.1: *Let $A, B \subseteq \mathbb{Z}^d$ and $\delta > 0$ and suppose that A is $(1 - \delta)$ -invariant to e_1, \dots, e_d , i.e.,*

$$|A \cap (A + e_i)| > (1 - \delta)|A|$$

for all $i = 1, \dots, d$. Then

$$|A + B| \geq (1 - 2d^2\delta)(|A|^{1/d} + |B|^{1/d})^d.$$

Proof: Let

$$\begin{aligned} A_0 &= \{u \in A : u + e_i \in A \text{ and } u + e_i + e_j \in A \text{ for all } i, j = 1, \dots, d\} \\ &= A \cap \bigcap_{1 \leq i \leq d} (A - e_i) \cap \bigcap_{1 \leq i < j \leq d} (A - e_i - e_j). \end{aligned}$$

From the invariance assumption, $A \cap (A - e_i)$ and $A \cap (A - e_i - e_j)$ are of size at least $(1 - 2\delta)|A|$, so

$$(2.1) \quad |A_0| > (1 - 2d^2\delta)|A|.$$

Consider now the sets $\tilde{A}_0, \tilde{B} \subseteq \mathbb{R}^d$ obtained by “thickening” A_0, B :

$$\tilde{A}_0 = A_0 + [0, 1)^d, \quad \tilde{B} = B + [0, 1)^d$$

(we identify A_0, B with subsets of \mathbb{R}^d in the obvious way). Clearly

$$\nu(\tilde{A}_0) = |A_0|, \quad \nu(\tilde{B}) = |B|,$$

so by the Brunn–Minkowsky inequality and equation (2.1),

$$\begin{aligned} \nu(\tilde{A}_0 + \tilde{B}) &\geq (|A_0|^{1/d} + |B|^{1/d})^d \\ &\geq (1 - 2d^2\delta)(|A|^{1/d} + |B|^{1/d})^d. \end{aligned}$$

On the other hand, it is a simple consequence of the definition of A_0 that if $u \in \mathbb{Z}^d \cap (\tilde{A}_0 + \tilde{B})$ then $u \in A + B$, and therefore

$$\tilde{A}_0 + \tilde{B} \subseteq (A + B) + [0, 1)^d,$$

so

$$\nu(\tilde{A}_0 + \tilde{B}) \leq |A + B|;$$

putting this all together, we get

$$|A + B| \geq (1 - 2d^2\delta)(|A|^{1/d} + |B|^{1/d})^d. \quad \blacksquare$$

One corollary of this is that in \mathbb{Z}^d , every Tempel’man sequence must have constant at least 2^d .

3. Tempel’man sequences and abelian rank

Suppose G is a countable group and contains elements e_1, \dots, e_d which generate a subgroup isomorphic to \mathbb{Z}^d . Then Theorem 2.1 can be extended to products of sufficiently invariant sets in G , where invariance is measured relative to e_1, \dots, e_d .

We first assume that one of the sets is contained in this abelian subgroup:

LEMMA 3.1: *Let G be a group and $H \cong \mathbb{Z}^d < G$. Let $A \subseteq H$ be $(1-\delta)$ -invariant with respect to the standard generators $e_1, \dots, e_d \in H$, i.e., $|(e_i + A) \cap A| \geq (1 - \delta)|A|$ for $i = 1, \dots, d$. Then for any finite $B \subseteq G$,*

$$|BA|, |AB| \geq (1 - 2d^2\delta)(|A|^{1/d} + |B|^{1/d})^d.$$

Proof: We show the inequality for $|AB|$. Let $\{Hg_i\}$ be an enumeration of the right cosets of H and set $B_i = B \cap Hg_i$. Using Theorem 2.1,

$$|AB| = \sum_i |AB_i| \geq (1 - 2d^2\delta) \sum_i (|A|^{1/d} + |B_i|^{1/d})^d$$

using the fact that $t \mapsto (c + t^{1/d})^d$ is concave and homogeneous and that $\sum_i |B_i| = |B|$,

$$\geq (1 - 2d^2\delta)(|A|^{1/d} + |B|^{1/d})^d. \quad \blacksquare$$

Next we drop the requirement that the invariant set be contained in the abelian subgroup, and instead consider products of the form $F^{-1}F$:

LEMMA 3.2: Suppose G is a group, $H \cong \mathbb{Z}^d < G$ with e_1, \dots, e_d the standard generators in H , and $F \subseteq G$ finite such that F is $(1 - \delta)$ -invariant with respect to left multiplication by e_1, \dots, e_d , i.e., $\frac{1}{|F|}|e_i F \cap F| \geq (1 - \delta)|F|$. Then

$$|F^{-1}F| \geq 2^d(1 - 2d^2\sqrt{\delta})(1 - d\sqrt{\delta})|F|.$$

Proof: Let Hg_i be an enumeration of the right cosets of H and $F_i = F \cap Hg_i$. From the invariance assumption there is a set of indices I such that

$$\left| \bigcup_{i \in I} F_i \right| \geq (1 - d\sqrt{\delta})|F|$$

and F_i is $(1 - \sqrt{\delta})$ -invariant to e_1, \dots, e_d from the left for each $i \in I$. Let $i_0 \in I$ be such that $|F_{i_0}|$ is maximal among $\{|F_i|\}_{i \in I}$. Now

$$|F^{-1}F| \geq |F_{i_0}^{-1}F| \geq \sum_{i \in I} |F_{i_0}^{-1}F_i|$$

by Lemma 3.1,

$$\geq \sum_{i \in I} (1 - 2d^2\sqrt{\delta})(|F_{i_0}|^{1/d} + |F_i|^{1/d})^d$$

by choice of i_0

$$\geq (1 - 2d^2\sqrt{\delta}) \sum_{i \in I} 2^d |F_i| \geq 2^d(1 - 2d^2\sqrt{\delta})(1 - d\sqrt{\delta})|F|. \quad \blacksquare$$

Recall that the abelian rank $r(G)$ of G is the largest n such that \mathbb{Z}^d can be embedded in G (Definition 1.2). Recall also the definition of Tempel'man sequences (Definition 1.1). As an immediate corollary of Lemma 3.2 we have

THEOREM 3.3: *If $r(G) < \infty$, then $2^{r(G)}$ is a lower bound on the constant associated with Tempel'man sequences in G .*

For abelian groups, there is a converse to this, which together with Theorem 3.5 characterizes those abelian groups which have Tempel'man sequences. I thank B. Weiss for suggesting the following construction:

THEOREM 3.4: *If A is a countable abelian group with $r(A) < \infty$, then there are Tempel'man sequences in A .*

Proof: Suppose that the rank of A is $d < \infty$. We will construct a Tempel'man sequence. Let $\{a_1, a_2, \dots\}$ be an enumeration of the elements of A . Let A_n be the group generated by $\{a_1, \dots, a_n\}$. Since A_n is finitely generated, we can write it as $T_n \oplus \mathbb{Z}^{d(n)}$ with T_n a finite group and $d(n) \leq d$. Set $F_n = T_n \times \{0, \dots, k(n)\}^{d(n)}$, where $k(n)$ is chosen large enough that F_n is $\frac{1}{n}$ -invariant to a_1, \dots, a_n and $F_{n-1} \subseteq F_n$ (it is easy to check that such a $k(n)$ exists). Now one verifies that $|F_n - F_n| \leq 2^{d(n)}|F_n|$. Thus $\{F_n\}$ is a Tempel'man sequence. ■

From the proof one also sees that if $r(A) = d < \infty$ then there exist Tempel'man sequences with constant 2^d , and Lemma 3.2 shows that is the best possible.

On the other hand, as noted in the introduction, finite rank in general does not guarantee the existence of Tempel'man sequences: the lamplighter group L studied in [4] has $r(L) = 1$ but has no Tempel'man sequences.

From Lemma 3.2 we also derive

THEOREM 3.5: *If G is an amenable group with $r(G) = \infty$, then G does not have Tempel'man sequences.*

4. Growth of tempered sequences

The previous section and the work of Lindenstrauss on the lamplighter group [4] show that there are amenable groups without Tempel'man sequences. However, as noted in the introduction, all amenable groups possess tempered sequences (Definition 1.3), which can serve as averaging sequences. Ideally, we would like there to be averaging sequences which grow slowly (in \mathbb{Z} they can grow linearly). It is therefore of interest to find conditions indicating or precluding such slow growth.

In order to analyze the growth rate of tempered sequences we will need information about the product of different sets, each of which is somewhat invariant:

LEMMA 4.1: Suppose G is a group, $H \cong \mathbb{Z}^d < G$ with e_1, \dots, e_d the standard generators in H , and $F_1, F_2 \subseteq G$ finite such that F_1, F_2 are $(1 - \delta)$ -invariant with respect to left-multiplication by e_1, \dots, e_d , i.e., $|e_i F_k \cap F_k| \geq (1 - \delta)|F_k|$ for $i = 1, \dots, d$ and $k = 1, 2$. Then

$$|F_1^{-1} F_2| \geq 2^d (1 - 2d^2 \sqrt{\delta})(1 - d\sqrt{\delta}) \min\{|F_1|, |F_2|\}.$$

Proof: Similar to Lemma 3.2. Let Hg_i be an enumeration of the right cosets of H and $F_{1,i} = F_1 \cap Hg_i$ and similarly $F_{2,i}$. From the invariance assumption there are sets of indices I, J such that

$$\left| \bigcup_{i \in I} F_{1,i} \right| \geq (1 - d\sqrt{\delta})|F_1|, \quad \left| \bigcup_{j \in J} F_{2,j} \right| \geq (1 - d\sqrt{\delta})|F_2|$$

and $F_{k,i}$ is $(1 - \sqrt{\delta})$ -invariant with respect to left multiplication by e_1, \dots, e_d for each $i \in I$ and $k = 1, 2$. Let $i_0 \in I$ be such that $|F_{1,i_0}|$ is maximal among $\{|F_{1,i}|\}_{i \in I}$ and $j_0 \in J$ such that $|F_{2,j_0}|$ is maximal among $\{|F_{2,j}|\}_{j \in J}$. Now the calculation in Lemma 3.2 shows that if $|F_{1,i_0}| \geq |F_{2,j_0}|$ then

$$|F_1^{-1} F_2| \geq 2^d (1 - 2d^2 \sqrt{\delta})(1 - d\sqrt{\delta})|F_2|,$$

while if $|F_{2,j_0}| \geq |F_{1,i_0}|$ then

$$|F_1^{-1} F_2| \geq 2^d (1 - 2d^2 \sqrt{\delta})(1 - d\sqrt{\delta})|F_1|,$$

which together complete the lemma. \blacksquare

THEOREM 4.2: If G is amenable and $r(G) = \infty$, then every tempered sequence in G grows super-exponentially.

Proof: Let $\{F_n\}$ be a tempered Følner sequence in G , so for some constant C , $|F_{n-1}^{-1} F_n| \leq C|F_n|$ for all n . Fix $d > 2 + \log_2 C$ and $\delta = 1/(16d^2)$. Let $H \cong \mathbb{Z}^d < G$ be generated by e_1, \dots, e_d . For large enough n the F_n 's will be $(1 - \delta)$ -invariant to e_1, \dots, e_d , so by Lemma 4.1 for large enough n we will have

$$C|F_n| \geq |F_{n-1}^{-1} F_n| \geq 2^d (1 - 2d^2 \sqrt{\delta})(1 - d\sqrt{\delta}) \min\{|F_n|, |F_{n-1}|\}.$$

substituting $\delta = 1/(16d^2)$ and rearranging gives

$$|F_n| \geq \frac{2^{d-2}}{C} \min\{|F_n|, |F_{n-1}|\}.$$

Since $d > 2 + \log_2 C$, this relation excludes the possibility that $\min\{|F_{n-1}|, |F_n|\} = |F_n|$, so it must be that

$$|F_n| \geq \frac{2^{d-2}}{C} |F_{n-1}|.$$

Since d was arbitrary and this holds for all large enough n , the proof is complete. ■

The results presented in this paper do not settle the question of the character of averaging sequences in the class of nonabelian groups of finite rank. In this class we have examples of groups with good sequences (e.g., nonabelian groups with polynomial growth) and groups with bad sequences (e.g., the lamplighter group L). Lindenstrauss has speculated that for finitely generated groups exponential growth may force super-exponential growth of tempered sequences and the absence of Tempel'man sequences. The following example provides some further support for this. Note that unlike L , it is torsion-free.

Example: Let \mathbb{Z} act on $\bigoplus_{i=-\infty}^{\infty} \mathbb{Z}$ by shift to the left, i.e., for $n \in \mathbb{Z}$ and $u = (u_i) \in \bigoplus_{i=-\infty}^{\infty} \mathbb{Z}$ we define $(u^n)_i = u_{i+n}$. Let

$$G = \mathbb{Z} \ltimes \bigoplus_{i=-\infty}^{\infty} \mathbb{Z}$$

explicitly, $G = \{(n, u) \in \mathbb{Z} \times \bigoplus_{i=-\infty}^{\infty} \mathbb{Z}\}$ and $(n, u) \cdot (m, v) = (n + m, u^m + v)$.

This group is finitely generated (e.g., by $(0, (\dots, 0, 0, 1, 0, 0, \dots))$ and $(1, (\dots, 0, 0, 0, \dots))$) and is torsion-free. G is also amenable, since it is solvable [3]. Clearly $r(G) = \infty$, so by Theorem 4.2, G has no Tempel'man sequences and every tempered sequence in G grows super-exponentially.

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